NORTH-HOLLAND

Constructive Characterization of Lipschitzian $\mathbf{Q}_0$-Matrices

G. S. R. Murthy
Indian Statistical Institute
110 Nelson Manickam Road
Aminjikarai
Madras 600 029, India

and

T. Parthasarathy and B. Sriparna
Indian Statistical Institute
7 SJS Sansanwal Marg
New Delhi 100 016, India

Submitted by Richard A. Brualdi

ABSTRACT

A matrix $M \in \mathbf{R}^{n \times n}$ has property $(**)$ if M and all its principal pivotal transforms (PPTs) satisfy the property that the rows corresponding to the nonpositive diagonal entries are nonpositive. It has been shown that every Lipschitzian $\mathbf{Q}_0$-matrix satisfies property $(**)$. In this paper, it is shown that property $(**)$ is also sufficient for a Lipschitzian matrix to be in $\mathbf{Q}_0$. Property $(**)$ has several consequences. If A has this property, then A and all its PPTs must be completely $\mathbf{Q}_0$; further, for any q, the linear complementarity problem (q, A) can be processed by a simple principal pivoting method. It is shown that a negative matrix is an $\mathbf{N}$-matrix if, and only if, it has property $(**)$; a matrix is a $\mathbf{P}$-matrix if, and only if, it has property $(**)$ and its value is positive. Property $(**)$ also yields a nice decomposition structure of Lipschitzian matrices. This paper also studies properties of Lipschitzian matrices in general; for example, we show that the Lipschitzian property is inherited by all the principal submatrices. © *Elsevier Science Inc., 1997*

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1. INTRODUCTION

The linear complementarity problem (LCP) with data $A \in \mathbf{R}^{n \times n}$ and $q \in \mathbf{R}^n$ is to find a vector $z \in \mathbf{R}^n$ such that

$$Az + q \geqslant 0, \qquad z \geqslant 0, \quad \text{and} \quad z^t(Az + q) = 0. \tag{1}$$

This problem is denoted by (q, A). Over the years, the LCP has emerged as a major field of research in mathematical programming and has found a wide variety of applications (see [3, 8, 10, 15, 16]). In connection with the study of this problem a number of matrix classes have been identified. Among them are $\mathbf{Q}$ and $\mathbf{Q}_0$ which are of fundamental interest [3, 14]. A number of algorithms have been developed to solve various instances of LCPs. Typically, each algorithm works for certain classes of problems engendered by the properties of the matrices. Our main goal of this paper is to characterize $\mathbf{Q}_0$-matrices within the class of Lipschitzian matrices and show that the simple principal pivoting method, due to Zoutendijk [18] and Bard [1], processes LCPs (q, A) when A is a Lipschitzian $\mathbf{Q}_0$-matrix. The above characterization is established in response to a conjecture raised by Murthy, Parthasarathy, and Sabatini [13] on Lipschitzian $\mathbf{Q}_0$-matrices. This states that if A is a Lipschitzian matrix, then it is in $\mathbf{Q}_0$ if, and only if, it satisfies property $(**)$ (see Definition 3.1). In this paper, we show that that is indeed the case, and establish that if A is a Lipschitzian matrix satisfying property $(**)$ and (q, A) has a feasible solution, then a solution to (q, A) can be obtained by using the above-mentioned principal pivoting method.

It will be seen that Property $(**)$ has several consequences. It is a sufficient condition for a matrix to be completely $\mathbf{Q}_0$. The characterization of completely $\mathbf{Q}_0$-matrices in general, envisaged by Cottle [2], has remained as an open problem (see also [4]). Murthy and Parthasarathy [11] have characterized the nonnegative completely $\mathbf{Q}_0$-matrices and also proved that symmetric copositive matrices are $\mathbf{Q}_0$ if, and only if, they are completely $\mathbf{Q}_0$. Property $(**)$ has another significance. It is shown that if A is a negative matrix, then it is an $\mathbf{N}$-matrix (that is, all the principal minors are negative) if, and only if, it satisfies property $(**)$. Also a matrix is in $\mathbf{P}$ if, and only if, it satisfies property $(**)$ and has positive value.

The motivation of our results in this paper comes from some conjectures on Lipschitzian matrices raised by Gowda [5, 6] and Pang. In general, it is difficult to decide whether a given matrix is Lipschitzian or not. Mangasarian and Shiau [9] proved that $\mathbf{P}$-matrices are Lipschitzian, and Gowda [6] showed that negative $\mathbf{N}$-matrices are Lipschitzian. Stone [17] asserts that the Lipschitzian matrices are just the INS matrices (see [3]). In Section 4, we derive some properties of Lipschitzian matrices. It is shown that if A is Lipschitzian, then so are the principal submatrices of A. Another result exhibits the decomposition structure of nonpositive Lipschitzian matrices (see Theorem 4.7).

2. PRELIMINARIES

We follow the notation adopted in [3]. Let $A \in \mathbf{R}^{n \times n}$ and $q \in \mathbf{R}^n$. Define the sets $F(q, A) = \{z \in \mathbf{R}^n_+ : Az + q \geqslant 0\}$ and $S(q, A) = \{z \in F(q, A) : (Az + q)^t z = 0\}$. A matrix A is said to be in matrix class $\mathbf{Q}$ if $S(q, A) \neq \varnothing$ for all $q \in \mathbf{R}^n$, and it is said to be in $\mathbf{Q}_0$ if $S(q, A) \neq \varnothing$ whenever $F(q, A) \neq \varnothing$. The matrix A is said to be completely $\mathbf{Q}$ (completely $\mathbf{Q}_0$) if $A_{\alpha\alpha}$, the principal submatrix of A with respect to α, is in $\mathbf{Q}$ ($\mathbf{Q}_0$) for all index sets α. If $A_{\alpha\alpha}$ is nonsingular, then the principal pivotal transform (PPT) of A with respect to α is defined as M where $M_{\alpha\alpha} = (A_{\alpha\alpha})^{-1}$, $M_{\alpha\bar{\alpha}} = -M_{\alpha\alpha} A_{\alpha\bar{\alpha}}$, $M_{\bar{\alpha}\alpha} = A_{\bar{\alpha}\alpha} M_{\alpha\alpha}$, and $M_{\bar{\alpha}\bar{\alpha}} = A_{\bar{\alpha}\bar{\alpha}} - M_{\bar{\alpha}\alpha} A_{\alpha\bar{\alpha}}$. By convention, A is a PPT of itself (here $\alpha = \varnothing$). Further, $\bar{q}$ defined by $\bar{q}_\alpha = -(A_{\alpha\alpha})^{-1} q_\alpha$, and $\bar{q}_{\bar{\alpha}} = q_{\bar{\alpha}} - (A_{\alpha\alpha})^{-1} q_\alpha$ is called the PPT of q with respect to A and α. The LCP $(\bar{q}, M)$ is called the PPT of (q, A) with respect to A and α. The PPTs are called *simple PPTs* if α is a singleton set.

One of the fundamental results on the LCP states that for any $A \in \mathbf{R}^{n \times n}$ and $q \in \mathbf{R}^n$, $F(q, A) \neq \varnothing$ ($S(q, A) \neq \varnothing$) if, and only if, $F(\bar{q}, M) \neq \varnothing$ $[S(\bar{q}, M) \neq \varnothing]$. Given (q, A), if there exists a PPT $(\bar{q}, M)$ of (q, A) in which $\bar{q} \geqslant 0$, then a solution to (q, A) can be easily obtained from that of $(\bar{q}, M)$. This is the basis of some of the algorithms to solve LCPs. Certain classes of LCPs can be processed by using only the simple PPTs. The algorithms which use only simple PPTs are known as simple principal pivoting methods or Bard-type methods (see [3] for details). We conclude this section with the following definition and a result due to Gowda [5].

DEFINITION 2.1. Say that $A \in \mathbf{R}^{n \times n}$ is a Lipschitzian matrix if the multivalued mapping Φ: $q \to S(q, A)$ satisfies the following property: there exists a positive constant C such that

$$S(q, A) \subseteq S(\bar{q}, A) + C\|q - \bar{q}\| B$$

holds for every q and $\bar{q}$ satisfying $S(q, A) \neq \varnothing$ and $S(\bar{q}, A) \neq \varnothing$. Here $B = \{x \in \mathbf{R}^n : \|x\| \leqslant 1\}$ is the closed unit ball in $\mathbf{R}^n$.

THEOREM 2.2. *Let A be a Lipschitzian matrix. Then every PPT of A is also Lipschitzian.*

3. LIPSCHITZIAN $\mathbf{Q}_0$-MATRICES

In [13], property $(**)$ defined below was obtained as a necessary condition on Lipschitzian $\mathbf{Q}_0$-matrices, and it was conjectured that the same is also sufficient for a Lipschitzian matrix to be in $\mathbf{Q}_0$. Besides settling this conjecture positively, it will be shown that property $(**)$ characterizes $\mathbf{N}$-matrices (of the second kind) and $\mathbf{P}$-matrices with additional assumptions.

DEFINITION 3.1. Let $A \in \mathbf{R}^{n \times n}$. Say that A satisfies *property* $(**)$ if every PPT M of A satisfies the condition that the rows corresponding to nonpositive diagonal entries of M are nonpositive.

REMARK 3.2. It should be noted that if A satisfies property $(**)$, then so does every principal submatrix of A. Also it is clear that property $(**)$ is satisfied by every PPT of A.

The following algorithm, due to Zoutendijk and Bard, is shown to work for a special class of LCPs (see Chapter 4 of [3]). This is a simple principal pivoting method in the sense that it transforms the given LCP into its PPTs using only the diagonal entries as the pivoting blocks in each iteration.

Consider the problem (q, A), where $A \in \mathbf{R}^{n \times n}$ and $q \in \mathbf{R}^n$. The following algorithm, taken from [3, p. 239], is presented in a different form with a specific choice of B.

ALGORITHM 3.3.

Step 0. *Initialization.* Set $M = A$, $p = q$, $B = I$ (the identity matrix in $\mathbf{R}^{n \times n}$), and $\alpha = \emptyset$.

Step 1. *Rule for termination.* If $p \geqslant 0$, then stop. The vector z defined by $z_\alpha = p_\alpha$ and $z_{\bar{\alpha}} = 0$ is a solution of (q, A).

Step 2. *Pivot selection.* Let r be the index such that

$$\frac{B_{r\cdot}}{q_r} = \text{lexico}\max\left\{\frac{B_{i\cdot}}{q_i} : q_i < 0\right\}.$$

Step 3. *Pivoting*. Replace p and the columns of B by their PPTs with respect to M and $\{r\}$. Replace M by the PPT of M with respect to $\{r\}$. Replace α by its symmetric difference with $\{r\}$. Go to step 1.

THEOREM 3.4. *Suppose $A \in \mathbf{R}^{n\times n}$ satisfies property $(**)$. Then A belongs to $\mathbf{Q}_0$.*

Proof. Let $q \in \mathbf{R}^n$ be such that $F(q, A) \neq \varnothing$. We will show that if Algorithm 3.3 is applied to (q, A), then it will terminate in a finite number of steps with a solution to (q, A). From the remarks made in Section 2, it follows that $F(p, M) \neq \varnothing$ in each iteration of the algorithm. This implies that, at each iteration of the algorithm, if $p_i < 0$ for any i, then $M_{i.}$ must have a positive entry. By property $(**)$, $m_{ii} > 0$ for each i such that $p_i < 0$. From Lemma 4.2.3 and Theorem 4.2.4 of [3], it follows that the algorithm terminates in a finite number of steps with a solution to (q, A) (see also Remark 4.2.5 of [3]). ∎

COROLLARY 3.5. *Suppose $A \in \mathbf{R}^{n\times n}$ satisfies property $(**)$. Then A (and every PPT of A) is completely $\mathbf{Q}_0$.*

Proof. Follows from Remark 3.2. ∎

THEOREM 3.6. *Suppose $A \in \mathbf{R}^{n\times n}$ is a Lipschitzian matrix. Then the following conditions are equivalent*:

(i) $A \in \mathbf{Q}_0$.
(ii) *A satisfies property $(**)$.*
(iii) *A is completely $\mathbf{Q}_0$.*

Proof. (i) $\Rightarrow$ (ii) has already been proved in [13]. However, for the sake of completeness, we will briefly outline the proof. Since the Lipschitzian and $\mathbf{Q}_0$ properties are both invariant under PPT, it suffices to show that the rows corresponding to the nonpositive diagonal entries of A are nonpositive. Assume $a_{11} \leqslant 0$. Suppose $A_{1.}$ has a positive entry. For each positive integer k, define $p^k = (0, k, \dots, k)^t$ and $q^k = (-1/k, k, \dots, k)^t$. Since $A_{1.}$ has a positive entry, for all large k, $F(q^k, A) \neq \varnothing$ and hence $S(q^k, A) \neq \varnothing$. Since $0 \in S(p^k, A)$ for all k, by the Lipschitzian property of A, there exists a sequence $z^k \in S(q^k, A)$ such that $z^k \to 0$ as $k \to \infty$. This forces $z_j^k = 0$, $j = 2, \dots, n$, for all large k. But then $(Az^k)_1 + q_1^k < 0$ for all large k, which is a contradiction.

(ii) $\Rightarrow$ (iii) follows from Corollary 3.5.
(iii) $\Rightarrow$ (i) is obvious. ∎

We shall now illustrate Algorithm 3.3 applied to (q, A) where A is a Lipschitzian $\mathbf{Q}_0$-matrix.

EXAMPLE 3.7. Consider the matrix

$$A = \begin{bmatrix} 3 & 2 & 0 & -1 \\ -2 & -1 & -1 & -1 \\ 3 & 2 & 1 & -1 \\ 4 & 3 & 2 & 1 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} 0 \\ 1 \\ -2 \\ -2 \end{bmatrix}.$$

Note that A is Lipschitzian, as the PPT of A with respect to a_{22} is a negative $\mathbf{N}$-matrix. For the sake of convenience, we shall use the following tableau.

Initialization. $\alpha = \emptyset$:

BV	z_1	z_2	z_3	z_4	p	$B_{\cdot 1}$	$B_{\cdot 2}$	$B_{\cdot 3}$	$B_{\cdot 4}$
w_1	3	2	0	−1	0	1	0	0	0
w_2	−2	−1	−1	−1	1	0	1	0	0
w_3	3	2	1	−1	−2	0	0	1	0
w_4	4	3	2	1	−2	0	0	0	1

First iteration. $r = 4$:

BV	z_1	z_2	z_3	w_4	p	$B_{\cdot 1}$	$B_{\cdot 2}$	$B_{\cdot 3}$	$B_{\cdot 4}$
w_1	7	5	2	−1	−2	1	0	0	1
w_2	2	2	1	−1	−1	0	1	0	1
w_3	7	5	3	−1	−4	0	0	1	1
z_4	−4	−3	−2	1	2	0	0	0	−1

Second iteration. $r = 3$:

BV	z_1	z_2	w_3	w_4	p	$B_{\cdot 1}$	$B_{\cdot 2}$	$B_{\cdot 3}$	$B_{\cdot 4}$
w_1	$\frac{7}{3}$	$\frac{5}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	1	0	$-\frac{2}{3}$	$\frac{1}{3}$
w_2	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	0	1	$-\frac{1}{3}$	$\frac{2}{3}$
z_3	$-\frac{7}{3}$	$-\frac{5}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{4}{3}$	0	0	$-\frac{1}{3}$	$-\frac{1}{3}$
z_4	$\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	0	0	$\frac{2}{3}$	$-\frac{1}{3}$

$$\alpha = \{3, 4\}.$$

Third iteration. $r = 4$:

BV	z_1	z_2	w_3	z_4	p	$B_{\cdot 1}$	$B_{\cdot 2}$	$B_{\cdot 3}$	$B_{\cdot 4}$
w_1	3	2	0	-1	0	1	0	0	0
w_2	1	1	-1	-2	-1	0	1	1	0
z_3	-3	-2	1	1	2	0	0	-1	0
w_4	-2	-1	2	3	2	0	0	-2	1

$$\alpha = \{3\}.$$

Fourth iteration. $r = 2$:

BV	z_1	w_2	w_3	z_4	p	$B_{\cdot 1}$	$B_{\cdot 2}$	$B_{\cdot 3}$	$B_{\cdot 4}$
w_1	1	2	2	3	2	1	-2	-2	0
z_2	-1	1	1	2	1	0	-1	-1	0
z_3	-1	-2	-1	-3	0	0	2	1	0
w_4	-1	-1	1	1	1	0	1	-1	1

$$\alpha = \{2, 3\}.$$

It can be checked that $z = (0, 1, 0, 0)^t$ is a solution of (q, A).

Negative **N**-matrices are known as **N**-matrices of the second kind. The following theorem shows that property $(**)$ is a necessary and sufficient condition for a negative matrix to be in **N**.

THEOREM 3.8. *Let $A \in \mathbf{R}^{n \times n}$ be a negative matrix. Then A is an* **N**-*matrix if, and only if, it satisfies property* $(**)$.

Proof. Since a negative $\mathbf{N}$-matrix is Lipschitzian and is in $\mathbf{Q}_0$, the "only if" part follows from Theorem 3.6. We shall prove the "if" part by induction on n. Obviously the theorem holds for $n = 1$. Assume that the theorem is true for all matrices of order $n - 1$, $n > 1$. Let A be an $n \times n$ matrix satisfying property $(**)$. Write

$$A = \begin{bmatrix} B & b \\ a^t & a_{nn} \end{bmatrix}.$$

By the induction hypothesis, $B \in \mathbf{N}$. Let M be the PPT of A with respect to B. Then $m_{nn} = a_{nn} - a^t B^{-1} b$. Let $y^t = a^t B^{-1}$. Suppose $m_{nn} \leqslant 0$. By property $(**)$, then, $y \leqslant 0$. This in turn implies $a^t = y^t B \geqslant 0$, which contradicts that $a^t < 0$. Hence $m_{nn} > 0$. Since $\det A = m_{nn} \det B$, $\det A < 0$. ■

The next theorem characterizes the $\mathbf{P}$-matrices in terms of property $(**)$. The value of a matrix $A \in \mathbf{R}^{n \times n}$ is said to be positive if there exists a nonnegative vector $x \in \mathbf{R}^n$ such that $Ax > 0$. The class of matrices with positive value is known as $\mathbf{S}$.

THEOREM 3.9. *Suppose $A \in \mathbf{R}^{n \times n}$. Then A is a $\mathbf{P}$-matrix if, and only if, A satisfies property $(**)$ and has positive value.*

Proof. We shall prove the "if" part. Since the value of A is positive, every PPT of A must also have positive value. If the value of a matrix is positive, then it cannot have any nonpositive rows. Since A satisfies property $(**)$, it follows that the diagonal entries of A and all its PPTs must be positive. From Parsons's result (Theorem 2 of [16]) it follows that A is a $\mathbf{P}$-matrix. ■

COROLLARY 3.10. $\mathbf{S} \cap (**) \equiv \mathbf{P}$.

Recently Gowda and Sznajder [7] showed that Lipschitzian matrices are nondegenerate (see Theorem 4.2 for an alternative proof of this). So, from Theorem 3.7, we see that every Lipschitzian $\mathbf{Q}_0$-matrix is a nondegenerate matrix satisfying property $(**)$. The question that arises now is whether the converse is true. From the above theorem, we see that the converse is true in the case of matrices with positive value. The following theorem shows that it is also true in other special cases.

THEOREM 3.11. *Let $A \in \mathbf{R}^{n\times n}$ be a nondegenerate matrix satisfying property $(**)$. Then A is a Lipschitzian matrix provided either of the following conditions is satisfied:*

(i) $A \leqslant 0$,
(ii) $A \geqslant 0$.

Proof. Case (i) can be proved easily using Theorem 4.7 (see Section 4). Case (ii) follows from Theorem 3.9, as the value of A will be positive in this case. ∎

One can prove the above theorem when $n = 2$ without any additional assumptions on A. Another question that can be raised is: if a Lipschitzian matrix satisfies property $(**)$, can we say that either the matrix is a **P**-matrix or a PPT of it is a negative **N**-matrix? The answer to this question is no. The matrices

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$$

serve as counterexamples.

4. PROPERTIES OF LIPSCHITZIAN MATRICES

In this section, we shall derive a number of properties of Lipschitzian matrices. So far there is no finite characterization of Lipschitzian matrices. Towards this end, we have the following results which provide necessary conditions.

THEOREM 4.1. *Suppose $A \in \mathbf{R}^{n\times n}$ is a Lipschitzian matrix. Then for any permutation matrix P and any positive diagonal matrix D, the products PAP^t, AD, and DA are all Lipschitzian matrices.*

The above theorem states that the Lipschitzian property is invariant under principal rearrangements and positive row and column scaling. We omit the easy proof of this theorem. The next theorem shows that every Lipschitzian matrix is nondegenerate. Gowda and Sznajder [7] proved this using piecewise affine functions. We give an alternative proof.

THEOREM 4.2. *Suppose $A \in \mathbf{R}^{n\times n}$ is Lipschitzian. Then A is nondegenerate.*

Proof. If we show that A has no zero diagonal entries, then from Theorem 2.2 it follows that the diagonal entries of every PPT of A are nonzero. From Corollary 3.5 of [15] it follows, then, that A is nondegenerate. Suppose $a_{ii} = 0$ for some i, say $i = 1$. Choose $\lambda > 0$ such that $\lambda A_{.1} + e > 0$, where $e = (1, 1, \dots, 1)^t \in \mathbf{R}^n$. Let $q = (0, 1, 1, \dots, 1)^t$, and for each positive integer k, let $q^k = (1/k, 1, 1, \dots, 1)^t \in \mathbf{R}^n$. Since $a_{11} = 0$, $\bar{z} = (\lambda, 0, \dots, 0)^t \in S(q, A)$. Also $S(q^k, A) \neq \varnothing\ \forall k \geqslant 1$. Since A is Lipschitzian, there exists a sequence z^k such $z^k \in S(q^k, A)$ and $\|z^k - \bar{z}\| \to 0$ as $k \to \infty$. This implies for all large k, $z_1^k > 0$ and $z_j^k = 0$ for $j = 2, 3, \dots, n$. This contradicts that $z^k \in S(q^k, A)$ for all large k. Hence $a_{11} \neq 0$. The theorem follows. ∎

THEOREM 4.3. *Suppose $A \in \mathbf{R}^{n\times n}$ is Lipschitzian. Then A is completely Lipschitzian, that is, all principal submatrices of A are Lipschitzian.*

Proof. We will imitate the proof by Gowda [6], which he gave in the case of negative matrices. Let M be the leading principal submatrix of A of order $n - 1$. We will show that M is Lipschitzian. Fix $\bar{p}$ and $\bar{q}$ in $\mathbf{R}^{(n-1)}$ such that $S(\bar{p}, M)$ and $S(\bar{q}, M)$ are nonempty. We will show that

$$S(\bar{p}, M) \subseteq S(\bar{q}, M) + C\|\bar{p} - \bar{q}\|\bar{B}, \tag{2}$$

where C is the Lipschitzian constant with respect to A, and $\bar{B}$ is the closed unit ball in $\mathbf{R}^{(n-1)}$. Let $\bar{x} \in S(\bar{p}, M)$. For each positive integer m, define $p^m = (\bar{p}^t, m)^t$ and $q^m = (\bar{q}^t, m)^t$. Note that, for all large m,

$$x = \begin{pmatrix} \bar{x} \\ 0 \end{pmatrix} \in S(p^m, A). \tag{3}$$

This implies $S(p^m, A) \neq \varnothing$. Similarly, $S(q^m, A) \neq \varnothing$. Since A is Lipschitzian and $\|p^m - q^m\| = \|\bar{p} - \bar{q}\|$, we have, for all m sufficiently large,

$$S(p^m, A) \subseteq S(q^m, A) + C\|\bar{p} - \bar{q}\|B. \tag{4}$$

From (3) and (4), it follows that there exists a sequence $z^m \in S(q^m, A)$ such that $\|z^m - x\| \leqslant C\|\bar{p} - \bar{q}\|$ for all large m. This implies $(Az^{m_0})_n + m_0 > 0$ for some positive integer m_0, which in turn implies that $z_n^{m_0} = 0$. This implies

$$\bar{x} = \bar{z} + C\|\bar{p} - \bar{q}\|\bar{u} \qquad \text{for some} \quad \bar{u} \in \bar{B},$$

where $\bar{z} = (z_1^{m_0}, z_2^{m_0}, \dots, z_{n-1}^{m_0})^t \in S(\bar{q}, M)$. This completes the proof of (2). It follows that M is Lipschitzian. The rest of the proof of the theorem follows by induction. ∎

LEMMA 4.4. *Suppose $A \in \mathbf{R}^{2\times 2}$ has one of the following sign structures*:

(i) $\begin{bmatrix} - & - \\ \oplus & - \end{bmatrix}$,

(ii) $\begin{bmatrix} - & \oplus \\ - & - \end{bmatrix}$,

where $\oplus$ stands for $+$ or 0. Then A is not Lipschitzian.

Proof. Note that if A has sign structure given by (i) or (ii), then $A \in \mathbf{Q}_0$. It is easy to see that A violates property $(**)$ [in both cases (i) and (ii)]. From Theorem 3.6 it follows that A is not Lipschitzian. ∎

LEMMA 4.5. *Suppose $A \in \mathbf{R}^{3\times 3}$ is a Lipschitzian matrix. Then A cannot have the following sign structure*:

$$\begin{bmatrix} - & 0 & - \\ 0 & - & - \\ - & - & - \end{bmatrix}.$$

Proof. Suppose A has the above sign structure. Then the sign structure of the PPT of A with respect to $\alpha = \{2, 3\}$ (which exists by Theorem 4.2) is given by

$$\begin{bmatrix} - & + & - \\ - & + & - \\ + & - & + \end{bmatrix}$$

(note that $\det A_{\alpha\alpha} < 0$ by Theorem 3.8). Clearly A is a $\mathbf{Q}_0$-matrix which violates property $(**)$. This contradicts that A is a Lipschitzian matrix. It follows that A cannot have the sign structure mentioned in the theorem. ■

THEOREM 4.6. *Suppose $A \in \mathbf{R}^{n\times n}$ is a Lipschitzian matrix. If the diagonal entries of A are negative, then $A \leqslant 0$.*

Proof. Note that every 2×2 matrix with negative diagonal entries is a $\mathbf{Q}_0$-matrix. By property $(**)$, it follows that every 2×2 principal submatrix is nonpositive. Hence $A \leqslant 0$. ■

THEOREM 4.7. *Suppose $A \in \mathbf{R}^{n\times n}$ is a nonpositive Lipschitzian matrix. Then there exists a permutation matrix P such that*

$$PAP^t = \begin{bmatrix} N^1 & 0 & \cdots & 0 \\ 0 & N^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & N^k \end{bmatrix},$$

where each N^j is a negative **N**-*matrix.*

Proof. We shall prove this by induction on n. From Lemma 4.4, the theorem is true for $n = 1$ or 2. Assume that the theorem is true for all square matrices of order less than or equal to $n - 1$, $n > 2$. Let $\alpha = \{i : a_{in} < 0\}$. We will show that $A_{\alpha\alpha} < 0$, $A_{\alpha\bar{\alpha}} = 0$, and $A_{\bar{\alpha}\alpha} = 0$. Let $i, j \in \alpha$ and let $\beta = \{i, j, n\}$. Then a principal rearrangement of $A_{\beta\beta}$ will have the sign pattern

$$\begin{bmatrix} - & \ominus & - \\ \ominus & - & - \\ - & - & - \end{bmatrix}.$$

If $a_{ij} = 0$ or $a_{ji} = 0$, then by Lemma 4.4, $a_{ij} = a_{ji} = 0$. This contradicts Lemma 4.5. It follows that $A_{\alpha\alpha} < 0$. A similar argument will show that

$A_{\alpha\bar{\alpha}} = 0$ and $A_{\bar{\alpha}\alpha} = 0$. By the induction hypothesis, $A_{\bar{\alpha}\bar{\alpha}}$ has the desired structure, and the theorem follows. ∎

COROLLARY 4.8. *Suppose $A \in \mathbf{R}^{n \times n}$ is a Lipschitzian matrix. Then A has the following structure*:

$$A = \begin{bmatrix} A_{11} & 0 & \cdots & 0 & A_{1k} \\ 0 & A_{22} & \ddots & 0 & A_{2k} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & A_{(k-1)(k-1)} & A_{(k-1)k} \\ A_{k1} & A_{k2} & \cdots & A_{k(k-1)} & A_{kk} \end{bmatrix},$$

where A_{ii}, $i = 1, 2, \ldots, k - 1$, are negative **N**-*matrices, and diagonal entries of A_{kk} are positive.*

Proof. Follows from Theorem 4.6 and Theorem 4.7. ∎

5. CONCLUDING REMARKS

In this paper, we have emphasized the importance of property $(**)$, which arose naturally as a necessary condition for a Lipschitzian matrix to be a $\mathbf{Q}_0$-matrix. One of the important consequences is that if A (or any of its PPTs) is either a **P**-matrix or an **N**-matrix of the second kind or is a Lipschitzian $\mathbf{Q}_0$-matrix, then the corresponding LCPs can be processed by simple principal pivoting method with suitable apparatus to avoid cycling. Besides characterizing the **P**-matrices and **N**-matrices of the second kind (with additional assumptions), property $(**)$ plays a crucial role in determining the structure of Lipschitzian matrices in general.

An interesting question raised by the referee is what can we say if we use columns instead of rows in the definition of property $(**)$. Consider the examples

$$A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix}.$$

It can be verified that A is a non-$\mathbf{Q}_0$ Lipschitzian matrix satisfying property $(**)$ with respect to columns. On the other hand, B is a Lipschitzian $\mathbf{Q}_0$-matrix, but B does not satisfy property $(**)$ with respect to columns. Another interesting question is the following.

OPEN PROBLEM. If $A \in \mathbf{R}^{n \times n}$ is a nondegenerate matrix satisfying property $(**)$, then is it true that A is Lipschitzian?

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